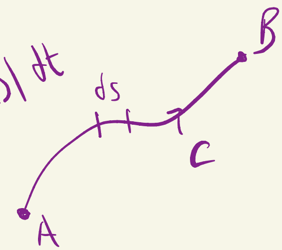
9-1

Recall: Arclength and line integrals

$$\text{Arc length}(C) = \int_C ds$$

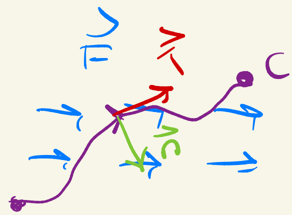
$ds = |\vec{r}\,'(t)|\,dt$



Line integral $f(x,y,z)$ along C

↑ scalar function!

$$\text{Line integral} = \int_C f(x,y,z)\,ds$$



Then, for "interesting" scalar functions using vector fields $\vec{F}$

$$\text{Work} = \int_C \vec{F}\cdot\vec{T}\,ds$$

(Flow)
($\oint$ circulation)

$\vec{F}$ - vector field

$\vec{T}$ - vector at each point on C (a form of vector field)

$\vec{F}\cdot\vec{T}$ = scalar function depending on coordinate (x,y,z)

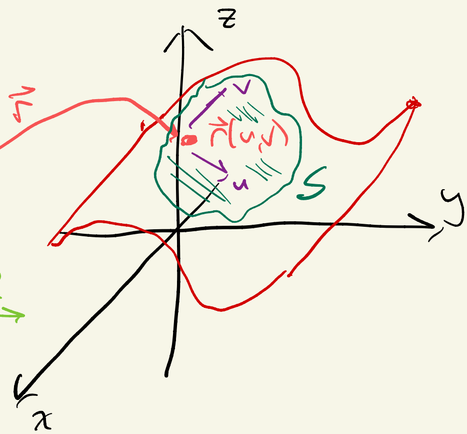
$$\text{Flux} = \int_C \vec{F}\cdot\vec{n}\,ds$$

Last time!

① Parameterize:

$$\vec{r}(u,v) = \overset{x}{f(u,v)}\,\hat{i} + \overset{y}{g(u,v)}\,\hat{j} + \overset{z}{h(u,v)}\,\hat{k}$$

$(u,v) \in R \subseteq \mathbb{R}^2$:

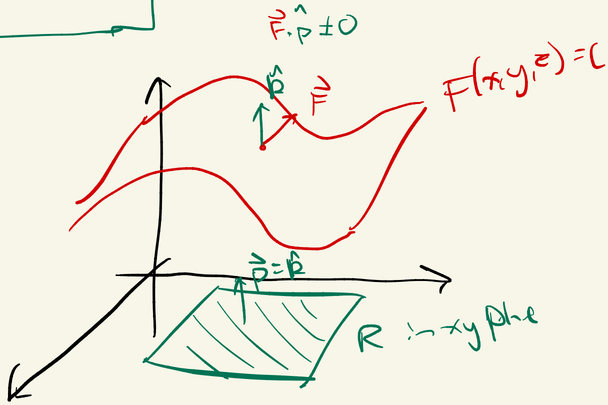


$S = \underbrace{\vec{r}(R)}_{\text{image of } R!}$

$d\sigma = \underbrace{|\vec{r}_u \times \vec{r}_v|}_{\text{"multidimensional" } \vec{r}\,'(t)}\,du\,dv$

$$\boxed{\text{Area} = \iint_S d\sigma = \iint_R |\vec{r}_u \times \vec{r}_v|\,du\,dv}$$

② Implicit Surface



$F(x,y,z)$ some scalar valued function. Their level sets are level surfaces $F(x,y,z) = c$

pick a plane whose normal $\hat{p}$ has $\nabla \vec{F}\cdot\hat{p} \neq 0$ over the entire region of integration

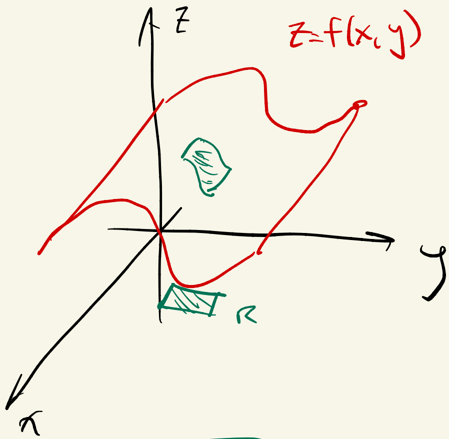
$$d\sigma = \frac{|\nabla F|}{|\nabla F\cdot\vec{p}|}\,dA$$

$$\boxed{\text{Area} = \iint_S d\sigma = \iint_R \frac{|\nabla F|}{|\nabla F\cdot\vec{p}|}\,dA}$$

③ Explicit Graph

$z = f(x,y)$.

$d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\ dA$



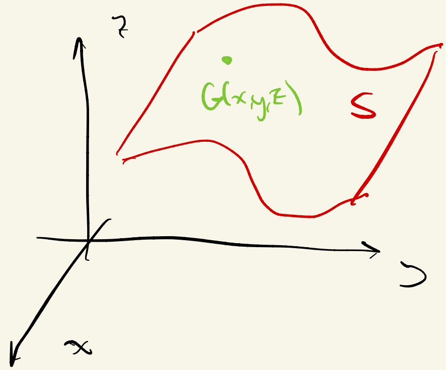
$$\boxed{\text{Area} = \iint_S d\sigma = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\ dx\,dy}$$

16.6

Now: How to define surface integrals?

Surface integrals

Let S be a smooth surface and $G(x,y,z)$ a scalar function defined on S



3 options:

① Parametric surface

$$\vec{r}(u,v) = \underbrace{f(u,v)}_{x}\,\hat{\imath} + \underbrace{g(u,v)}_{y}\,\hat{\jmath} + \underbrace{h(u,v)}_{z}\,\hat{k}$$

$$(u,v) \in R \subseteq \mathbb{R}^2$$

in uv plane

$$\iint_S G(x,y,z)\,d\sigma = \iint_R G(\vec{r}(u,v))\,\underbrace{|\vec{r}_u \times \vec{r}_v|\,du\,dv}_{d\sigma}$$

② Implicit surface

$F(x,y,z) = C$ defines a surface S above region R ← here, $R \subseteq xy$ plane or yz or xz

with $\hat{p}$ a unit vector normal to R such that $\nabla F \cdot \hat{p} \neq 0$ on R

$$\iint_S G(x,y,z)\,d\sigma = \iint_R G(x,y,z)\,\frac{|\nabla F|}{|\nabla F \cdot \hat{p}|}\,dA$$

③ Explicit surface

Graph $z = f(x,y)$ for $(x,y) \in R$

$$\iint_S G(x,y,z)\,d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

- Cool immediate applications
  - Scalar Functions on 3D: Pressure $= \frac{\text{Force}}{\text{Area}}$, ← could surface integrals to $\int \text{Pressure}\, dA = \text{Total force}$

    Temperature

    Concentration
  - Vector Fields in 3D: Fluid flow

    heat flow

    current

    magnetic flux

Ex:

Find $\iint_S x^2\, d\sigma$ for S the unit sphere.

Soln:

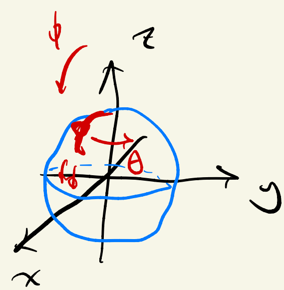
On E.C. Tuesday, we saw how to parameterize the sphere (radius = 1):

$$\vec{r}(\underset{u}{\phi}, \underset{v}{\theta}) = \underset{x}{\sin\phi\cos\theta}\,\hat{i} + \underset{y}{\sin\phi\sin\theta}\,\hat{j} + \underset{z}{\cos\phi}\,\hat{k}$$

$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}$

$$0 \leq \phi \leq \pi$$

$$0 \leq \theta \leq 2\pi$$



Calculated: $\rho^2 \sin\phi \to \rho = 1$ since unit sphere

$$|\vec{r}_u \times \vec{r}_v| = \sin\phi$$

$$\Rightarrow d\sigma = |\vec{r}_u \times \vec{r}_v|\, d\phi\, d\theta$$

$$= \sin\phi\, d\phi\, d\theta$$

$$\iint_S x^2\, d\sigma = \int_0^{2\pi}\int_0^{\pi} \underbrace{(\sin\phi\cos\theta)^2}_{x^2} \underbrace{\sin\phi\, d\phi\, d\theta}_{d\sigma}$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi \cos^2\theta \;\, d\phi\, d\theta$$

$$= \frac{4\pi}{3}$$

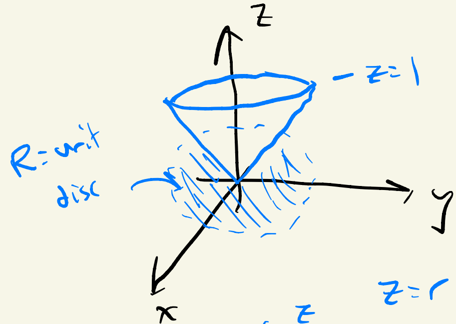
Ex:

Integrate $\iint_S x^2 \, d\sigma$ over S the cone

$z = \sqrt{x^2+y^2}$ $\quad 0 \le z \le 1$

Soln:

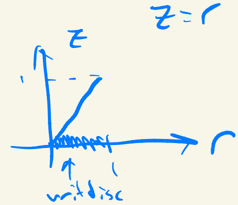
Explicit surface:



$$f(x,y) = \sqrt{x^2+y^2}$$

$$f_x = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}}$$



$$d\sigma = \sqrt{f_x^2 + f_y^2 + 1} \; dx\,dy$$

$$= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} \; dx\,dy$$

$$= \sqrt{1+1} \; dx\,dy$$

$$= \sqrt{2} \, dx\,dy$$

$$\Rightarrow \iint_S x^2 \, d\sigma = \iint_R x^2 \sqrt{2} \, dA$$

polar $$= \sqrt{2} \int_0^{2\pi} \int_0^1 (r\cos\theta)^2 r \, dr\, d\theta$$

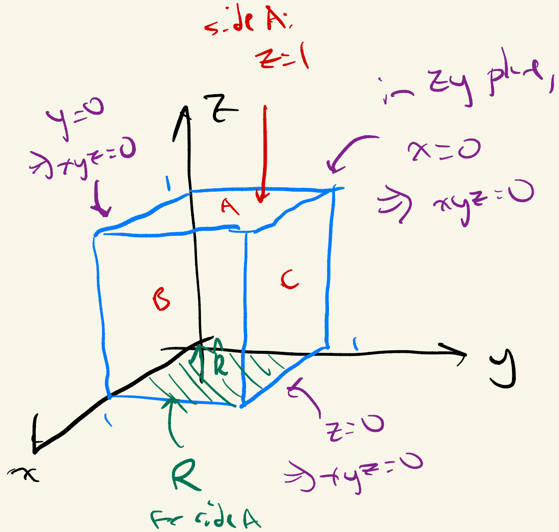
$$= \sqrt{2} \int_0^{2\pi} \int_0^1 r^3 \cos^2\theta \, dr\, d\theta$$

$$= \frac{\pi\sqrt{2}}{4}$$

Ex:

$\iint_S xyz\, d\sigma$ for S the surface of the unit cube in first octant

Soln:



$\iint_S xyz\, d\sigma =$

$\iint_A xyz\, d\sigma + \iint_B xyz\, d\sigma + \iint_C xyz\, d\sigma$

$+ \iint_{\text{Rest of cube}} 0\, d\sigma$

side A:

Let's do this implicitly (parameterization is really easy here)

$F(x,y,z) = z$

$z=1$ side A

$\nabla F = \hat{k}$. want to pick $\hat{p}$ so that $\nabla F \cdot \hat{p} \neq 0$ over integration region

$\Rightarrow$ pick $\hat{p} = \hat{k}$ $\Rightarrow$ integrate over $R \subseteq xy$ plane

$|\nabla F| = 1$

$|\nabla F \cdot \hat{p}| = |\hat{k} \cdot \hat{k}| = 1$

$\Rightarrow d\sigma = \dfrac{|\nabla F|}{|\nabla F \cdot \hat{p}|}\, dx\, dy = dx\, dy$

$$\iint_A xyz\, d\sigma \underset{\substack{z=1 \text{ on} \\ A}}{=} \int_0^1 \int_0^1 xy(1)\, dx\, dy$$

$$= \int_0^1 x \left[\tfrac{1}{2} y^2\right]_{y=0}^{y=1} dx$$

$$= \frac{1}{2} \int_0^1 x\, dx$$

$$= \frac{1}{4}$$

By symmetry (since integrating xyz over symmetric region)

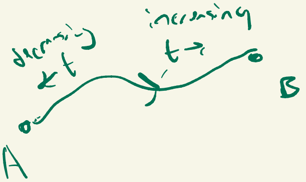
$$\frac{1}{4} = \iint_A xyz\, d\sigma = \iint_B xyz\, d\sigma = \iint_C xyz\, d\sigma$$

$$\Rightarrow \iint_S xyz\, d\sigma = \frac{3}{4}$$

OK, we can integrate scalar functions.

How to play with vector fields? (in other words, how to convert vector fields into interesting scalar functions using dot product and vector fields ~~for our~~ surface?)

Recall: to do line integrals, we needed a notion of orientation for our curves C

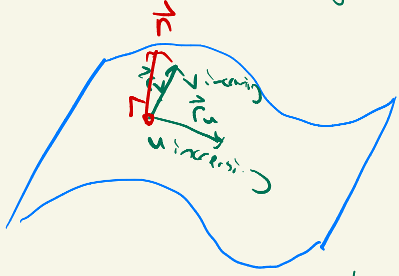


choice of parametrization "direction" = decides orientation

How do we orient surfaces?

we need to label which way u is increasing and which way v is increasing.

But for 2D surfaces, we can get away with less:



- if parametrized by (u,v), right hand rule tells us:

$$\vec{r}_u \times \vec{r}_v = \vec{n}$$

choose $\vec{n}$ to be a unit vector

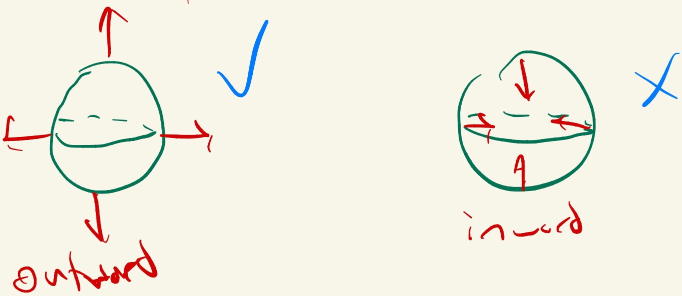
Think: $\left(\hat{i} \times \hat{j} = \hat{k}\right.$

$\vec{n}$'s direction labels "top" and "bottom"

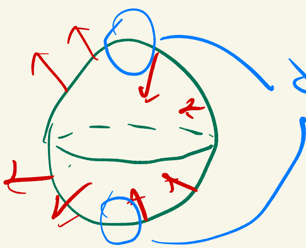
faces in $\vec{n}$ direction — faces in $-\vec{n}$ direction

For surfaces: picking a normal vector direction = picking an orientation.

Focus: pick outward normal for orientation of closed surfaces



· Choice of unit normal has to be continuous:



discontinuities in "normal vector field" always (vector field that assigns to each point s on the surface a normal vector)

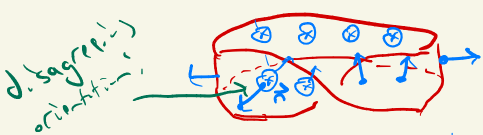
Illegal!

Warning: Topology

Not all 2D surfaces are orientable! (continuous choice of normal vector)

There are surfaces where we can't pick a normal vector field

Möbius Strip:



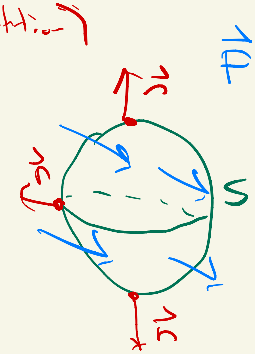
No continuous normal vector field on this surface

$\Rightarrow$ no "inside" or "outside"
"top" or "bottom"

Focus: only consider orientable surfaces S.

Def: Flux Integrals

$\vec F$ is a vector field, S a smooth oriented surface with unit normal vector field $\vec n$ (i.e. we've chosen orientation)

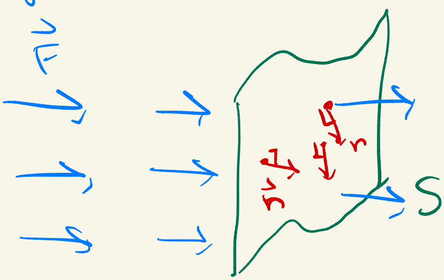


Then

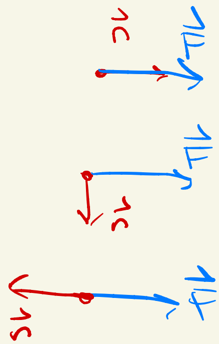
Flux integral of $\vec F$ through S $= \iint_S \vec F \cdot \vec n \, d\sigma$

(aka surface integral)

Again, just like flux integrals for curves, imagine a sail:



at each point: $\vec F \cdot \vec n$ is how much wind "hits" the sail



$\vec F \cdot \vec n > 0$

$\vec F \cdot \vec n = 0$

$\vec F \cdot \vec n < 0$

Ex:

Find the flux of $\vec{F} = \hat{i} + yz\hat{j} + z^2\hat{k}$

through S cut from the cylinder

$y^2+z^2=1$, $z \geq 0$, oriented outward

$0 \leq x \leq 1$

Soln:

· Find S and $d\sigma$.

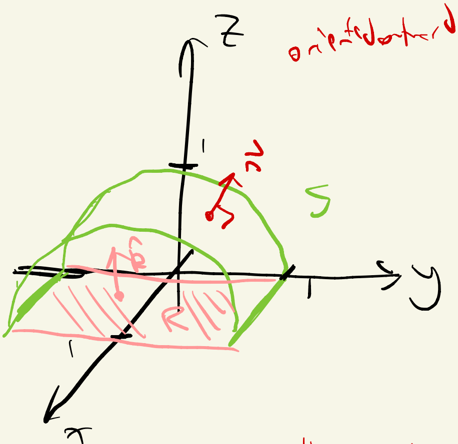
Implicit is a natural choice:

$G(x,y,z) = y^2+z^2$

level surface is $y^2+z^2=1$

enforce: $z \geq 0$

$0 \leq x \leq 1$



$\nabla G = 0\hat{i} + 2y\hat{j} + 2z\hat{k}$ ⟵ since ∇G orthogonal to level set of S,

$\vec{n} = \pm \frac{\nabla G}{|\nabla G|}$

orientation

unit vector

want: $\nabla G \cdot \hat{p} \neq 0$

$z=0$ only at boundary, meaning no contribution to $\iint_S d\sigma$.

$\Rightarrow$ choose $\hat{p} = \hat{k}$

$y^2+z^2=1$, $z \geq 0$

$$\Rightarrow \quad d\sigma = \frac{|\nabla G|}{|\nabla G \cdot \hat{p}|}\, dA = \frac{\sqrt{4y^2+4z^2}}{\sqrt{4z^2}}\, dA = \frac{2}{2z}\, dA$$

$\nabla G \cdot \hat{p} = \nabla G \cdot \hat{k}$

$= 2z$

• Need $\vec{n}$

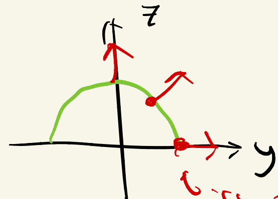
we already said:

$$\vec{n} = \pm \frac{\nabla G}{|\nabla G|} = \pm \left(\frac{2y\,\hat{j} + 2z\,\hat{k}}{2\sqrt{y^2+z^2}} \right)$$

$$\overset{y^2+z^2=1}{=} \pm \left(y\,\hat{j} + z\,\hat{k} \right)$$

pick it so it's outfacing:

$\Rightarrow$ pick $+$



if $(y,z)=(1,0)$, want $\vec{n}=\hat{j}$

$$\Rightarrow \text{Flux} = \iint_S \vec{F} \cdot \hat{n}\, d\sigma$$

$$= \iint_R \left(\hat{i} + yz\,\hat{j} + z\,\hat{k}\right) \cdot \left(y\,\hat{j} + z\,\hat{k}\right) d\sigma$$

$$= \int_0^1 \int_{-1}^1 y^2 z + z^3 \underbrace{\left(\frac{2}{2z}\right) dx\,dy}_{d\sigma}$$

notice - not need to find z in terms of x, y

$$= \int_0^1 \int_{-1}^1 \underbrace{(y^2+z^2)}_{1}\, \cancel{z} \left(\frac{\cancel{2}}{\cancel{2z}}\right) dx\,dy$$

$$= \int_0^1 \int_{-1}^1 dx\,dy$$

$$= 2$$

$\blacksquare$